Group Report 1964-11

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30 January 1964

Prepared under Electronic Systems Division Contract AF 19(628)-500 by

Lincoln Laboratory
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Lexington, Massachusetts

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the U.S. Air Force under Contract AF 19(628)-500.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

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ABSTRACT

This paper contains an analysis of the errors introduced into a monopulse radar system by employment of a digital matched filter containing two high speed analog-to-digital converters and describes errors predicted by a computer simulation of the system. The analysis and simulation indicates that a digital monopulse would compare very favorably in performance with an analog monopulse system.

This technical documentary report is approved for distribution.

Franklin C. Hudson
Franklin C. Hudson, Deputy Chief
Air Force Lincoln Laboratory Office

* The digital monopulse radar was proposed and designed by E. Gehrels, G. M. Hyde, L. G. Kraft, and J. Arthur, Group 31, Lincoln Laboratory.

1. INTRODUCTION

A block diagram of the Millstone monopulse radar control system is shown in Fig. 1. Azimuth and elevation sum and difference signals (denoted by Δ_{EL}, Δ_{AZ}, $\Sigma_{Ortho.}$, and $\Sigma_{Dir.}$) of frequency 1295mc are formed at the antenna from the four received signals. The four sum and difference frequencies are beat down to 28 mc, 29mc, 30mc, and 31mc respectively near the receiving antenna and then linearly combined and transmitted to the ground through a single channel. The doppler is subtracted out and the signal is again beat down by mixing it with a 30mc signal. The sine and cosine components of the resulting signal are then separated and assuming that they have an additive noise component the resulting two signals have the form given in Eqs. (1) - (2).

$$y_1 = S\cos 2\pi f_1 t - T\sin 2\pi f_1 t + U\cos 2\pi f_1 t + V\sin 2\pi f_1 t + X\cos 2\pi f_2 t + Y\sin 2\pi f_2 t + P + \text{Noise} \qquad (1)$$

$$y_2 = S\sin 2\pi f_1 t + T\cos 2\pi f_1 t - U\sin 2\pi f_1 t + V\cos 2\pi f_1 t - X\sin 2\pi f_2 t + Y\cos 2\pi f_2 t - Q + \text{Noise} \qquad (2)$$

where

$$f_1 = 10^6 \text{ cycles/sec.}$$

$$f_2 = 2 \cdot 10^6 \text{ cycles/sec.}$$

and the coordinate pairs (S, T), (U, V), (X, Y), and (P, Q) are the complex coordinates of vectors defined as follows.

$$\Delta_{EL} = \text{Re}\,[\,(S+iT)\,e^{-iwt}] = S\sin wt + T\cos wt$$

$$\Delta_{AZ} = \text{Re}[\,(U+iV)\,e^{-iwt}] = U\sin wt + V\cos wt$$

$$\Sigma_{Ortho.} = \text{Re}\,[\,(X+iY)\,e^{-iwt}] = X\sin wt + Y\cos wt$$

$$\Sigma_{Dir.} = \text{Re}\,[\,(P+iQ)\,e^{-iwt}] = P\sin wt + Q\cos wt$$

A digital filter is employed to determine S, T, U, V, X, Y, P, and Q by separating out the various components of y_1 and y_2 in the following manner. A number, m, of uniformally spaced samples of y_1 and y_2 one-fourth of a usec apart ($\frac{1}{4}$ of a cycle with respect to the 1mc components of y_1 and y_2) are added together as specified by Eqs. (3) - (10). The time of duration of the sampled pulse is nominally 2 millisec. so that a total of 8,000 samples (m = 8,000) are added together in accordance with Eqs. (3) - (10). The resulting sums are proportional to the appropriate components of y_1 and y_2 under

$$\tilde{S} = \sum_{n=0}^{1999} \frac{y_1\left(2n\pi\right) + y_2\left((2n+\frac{1}{2})\pi\right) - y_1\left((2n+1)\pi\right) - y_2\left((2n+\frac{3}{2})\pi\right)}{4 \cdot 2000} \tag{3}$$

$$\tilde{T} = \sum_{n=0}^{1999} \frac{y_2\left(2n\pi\right) - y_1\left((2n+\frac{1}{2})\pi\right) - y_2\left((2n+1)\pi\right) + y_1\left((2n+\frac{3}{2})\pi\right)}{4 \cdot 2000} \tag{4}$$

$$\tilde{U} = \sum_{n=0}^{1999} \frac{y_1\left((2n\pi)\right) - y_2\left((2n+\frac{1}{2})\pi\right) - y_1\left((2n+1)\pi\right) + y_2\left((2n+\frac{3}{2})\pi\right)}{4 \cdot 2000} \tag{5}$$

$$\tilde{V} = \sum_{n=0}^{1999} \frac{y_2\left(2n\pi\right) + y_1\left((2n+\frac{1}{2})\pi\right) - y_2\left((2n+1)\pi\right) - y_1\left((2n+\frac{3}{2})\pi\right)}{4 \cdot 2000} \tag{6}$$

$$\tilde{X} = \sum_{n=0}^{1999} \frac{y_1\left(2n\pi\right) - y_1\left((2n+\frac{1}{2})\pi\right) + y_1\left((2n+1)\pi\right) - y_1\left((2n+\frac{3}{2})\pi\right)}{2 \cdot 4000} \tag{7}$$

$$\tilde{Y} = \sum_{n=0}^{1999} \frac{y_2\left(2n\pi\right) - y_2\left((2n+\frac{1}{2})\pi\right) + y_2\left((2n+1)\pi\right) - y_2\left((2n+\frac{3}{2})\pi\right)}{2 \cdot 4000} \tag{8}$$

$$\tilde{P} = \sum_{n=0}^{1999} \frac{y_1\left(2n\pi\right) + y_1\left((2n+\frac{1}{2})\pi\right) + y_1\left((2n+1)\pi\right) + y_1\left((2n+\frac{3}{2})\pi\right)}{2 \cdot 4000} \tag{9}$$

$$\tilde{Q} = \sum_{n=0}^{1999} \frac{-y_2\left(2n\pi\right) - y_2\left((2n+\frac{1}{2})\pi\right) - y_2\left((2n+1)\pi\right) - y_2\left((2n+\frac{3}{2})\pi\right)}{2 \cdot 4000} \tag{10}$$

(the symbol ~ indicates the estimate of the signal)

the assumption that the mean of m samples of noise is zero. For example, it is seen that evaluation of the expression specified in Eq. (3) results in a signal proportional to S provided that the mean of the noise samples is zero.

$$\tilde{S} = \sum_{n=0}^{1999} \frac{y_1\left(2n\pi\right) + y_2\left((2n+\frac{1}{2})\pi\right) - y_2\left((2n+1)\pi\right) - y_2\left((2n+\frac{3}{2})\pi\right)}{4 \cdot 2000} \tag{3}$$

where

$$\begin{aligned}
y_1\left(2n\pi\right) &= S + U + X \quad + P \quad + \text{Noise} \\
y_2\left((2n+\tfrac{1}{2})\pi\right) &= S - U - Y \quad - Q \quad + \text{Noise} \\
-y_1\left((2n+1)\pi\right) &= S + U - X \quad - P \quad + \text{Noise} \\
-y_2\left((2n+\tfrac{3}{2})\pi\right) &= S - U + Y \quad + Q \quad + \text{Noise}
\end{aligned}$$

$$\tilde{S} = S + \frac{\sum_{i=0}^{7999} (\text{Noise})_i}{4 \cdot 2000}$$

The antenna azimuth and elevation control signals may be obtained from the estimates for S, T, U, V, X, and Y by computing the normalized projections of the vectors specified by the coordinate pairs (S, T), and (U, V) on the vector specified by the coordinate pair (X, Y) as presented in Eqs. (11) - (12). A subsequent paper will derive these equations and describe the computer necessary to perform these calculations.

$$\text{Azimuth signal to servo} = \frac{\Delta_{AZ}}{\Sigma_{Ortho}} = \frac{UX + VY}{X^2 + Y^2} \tag{11}$$

$$\text{Elevation signal to servo} = \frac{\Delta_{EL}}{\Sigma_{Ortho}} = \frac{SX + TY}{X^2 + Y^2} \tag{12}$$

2. ERROR ANALYSIS

An analysis of the errors in the estimates of the amplitude of the components, S, T, U, V, X, Y, P, and Q, of the two noisy signals, y_1 and y_2 is now examined. The signals are of the form given in Eqs. (1) - (2). Since the signals are sampled synchronously at a time when $2\pi f_1 t = \frac{n\pi}{2}$ (n integral) the signals can be considered to be of the form

$$y(t_i) = S + n(t_i) \tag{13}$$

where $n(t_i)$ is the magnitude of the noise at the time of the i^{th} sample and S is the component of y_1 and y_2 that is to be estimated. If the value of the i^{th} quantized sample is denoted by $x(t_i)$, then the estimate, $\widetilde{S}$, of the noisy signal is

$$\tilde{S} = \frac{1}{m} \sum_{i=1}^{m} x(t_i). \tag{14}$$

The noise is assumed to be additive uncorrelated from sample to sample, stationary and normally distributed with zero mean and variance, σ_n^2.

$$p\left(n(t_i)\right) = (2\pi\sigma_n^2)^{-1/2} \quad \exp\left\{-\frac{1}{2} n^2(t_i) / \sigma_n^2\right\} \tag{15}$$

In general, a quantizer has the input-output characteristic shown in Fig. 2. Thus, if the input to the quantizer, $y(t_i)$ is such that $b_i \leq y(t_i) < b_{i+1}$ then the output of the quantizer, $x(t_i)$, is a_i where $b_i \leq a_i \leq b_{i+1}$ but a_i is not necessarily equal to $\frac{b_i + b_{i+1}}{2}$. It should be noted that the assumed quantizer is symmetric:

$$a_{-i} = -a_i$$

and

$$b_{-i} = -b_i,$$

but $(a_{i+k} - a_i)$ is not necessarily equal to $(a_{j+k} - a_j)$ nor is $(b_{i+k} - b_i)$ necessarily equal to $(b_{j+k} - b_j)$ (i.e., equal quantizing levels are not necessarily assumed). The quantizer input-output characteristic is completely determined by the two column vectors

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_\ell \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ b_2 \\ \vdots \\ b_\ell \end{bmatrix}$$

The vector A determines the ℓ output levels and in order to be easily realizable it must be of the form

$$A = a \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ \ell \end{bmatrix}$$

(equal quantizing steps) or of the form

$$A = a \begin{bmatrix} 2^{i_1} \\ 2^{i_2} \\ \vdots \\ 2^{i_\ell} \end{bmatrix}$$

where i_j is an integer (logarithmic quantizing steps).

The vector B determines the division of the input into various levels, and the components of B may take on any values without presenting significant implementation problems. (These levels are usually determined by the value of resistors).

The probability density of the quantizer output, x, is

$$p(x) = \sum_{\substack{k=-\ell \\ k \neq 0}}^{\ell} p_k \, \delta(x - a_k).$$

where

$$\begin{aligned} p_k &= \text{Prob. } [x = a_k] \\ &= \text{Prob. } [b_k < y < b_{k+1}] \\ &= \int_{b_k}^{b_{k+1}} (2\pi\sigma_n^2)^{-1/2} \exp\left\{ -\frac{1}{2} \frac{(y-s)^2}{\sigma_n^2} \right\} dy \end{aligned}$$

It is convenient to define the following column vectors:

$$P = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_\ell \end{bmatrix} \qquad Q = \begin{bmatrix} p_{-1} \\ p_{-2} \\ \vdots \\ p_{-\ell} \end{bmatrix} \qquad \text{and } A_2 = \begin{bmatrix} a_1^2 \\ a_2^2 \\ \vdots \\ a_\ell^2 \end{bmatrix}$$

The average value of the quantizer output $E[x]$ is then

$$E[x] = \sum_{k=-\ell}^{\ell} p_k a_k$$

$$= A^T(P - Q) \tag{16}$$

and the variance σ_x^2 is

$$\sigma_x^2 = \left[\sum_{k=-\ell}^{\ell} a_k^2 p_k\right] - \left[E[x]\right]^2$$

$$= A_2^T(P+Q) - [A^T(P-Q)]^2 \tag{17}$$

where A^T is the transpose of A, the vector defining the quantizer output levels and P and Q are functions of the vector B. Since the output of the sampler is independent (uncorrelated) from sample to sample, the output of the quantizer is also independent from sample to sample.

The sum,

$$\tilde{S} = \frac{1}{m} \sum_{i=1}^{m} x(t_i) \tag{14}$$

stored in the accumulator is used as an estimate of S. The average value of $\tilde{S}$ is

$$E[\tilde{S}] = E\left[\frac{1}{m} \sum_{i=1}^{m} x(t_i)\right] = \frac{1}{m} \sum_{i=1}^{m} E[x] = E[x]$$

Therefore the average value of the estimate of S is

$$E[\tilde{S}] = A^T(P-Q). \tag{18}$$

The variance of $\tilde{S}$ is

$$\sigma_{\tilde{S}}^2 = E\left[\tilde{S}^2\right] - \left(E\left[\tilde{S}\right]\right)^2$$

$$= E\left[\frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} x(t_i)x(t_j)\right] - \left[A^T(P-Q)\right]^2.$$

However

$$E[x(t_i)x(t_j)] = \begin{cases} E[x(t_i)]\ E[x(t_j)], & i \neq j \\ E[x^2(t_i)] & , \quad i = j \end{cases}$$

$$= \begin{cases} [A^T(P-Q)]^2, & i \neq j \\ A_2^T(P+Q), & i = j. \end{cases}$$

Therefore

$$\sigma_{\tilde{S}}^2 = \frac{1}{m^2}\left[mA_2^T(P+Q) + (m^2-m)[A^T(P-Q)]^2\right] - [A^T(P-Q)]^2$$

$$= \frac{1}{m}\left[A_2^T(P+Q) - [A^T(P-Q)]^2\right] \tag{19}$$

$$= \frac{1}{m}\sigma_x^2.$$

The relative error, r, is defined as

$$r = \frac{S - \tilde{S}}{S} . \tag{20}$$

The average value of this error, E[r], is

$$\begin{aligned} E[r] &= \frac{S - E[\tilde{S}]}{S} \\ &= \frac{S - A^T(P-Q)}{S} \end{aligned} \tag{21}$$

and the variance of the relative error is

$$\begin{aligned} \sigma_r^2 &= E[r^2] - E^2[r] \\ &= \frac{\sigma_{\tilde{S}}^2}{S^2} \\ &= \left(\frac{1}{m}\right)\left(\frac{\sigma_x^2}{S^2}\right) \\ &= \left(\frac{1}{mS^2}\right)\left(A_2^T (P+Q) - \left[A^T(P-Q)\right]^2\right) \end{aligned} \tag{22}$$

It is seen from Eq. (21) that the average value of the relative error resulting from employment of a digital filter is independent of the number of samples and dependent only on the characteristics of the quantizer. On the other hand, the variance of the relative error (Eq. (22)) is inversely proportional to the number of samples, m, and depends on both the number of samples and the characteristics of the quantizer. Therefore, increasing m, the number of samples, will decrease the variance of the relative error, but will have no effect upon the average value of the relative error.

For large m, the Central Limit Theorem may be invoked. Then $\tilde{S}$ is normally distributed with variance $\sigma^2_{\tilde{S}}$ and mean $E[\tilde{S}]$. Thus for large m

$$p(\tilde{S}) = \left(2\pi\sigma^2_{\tilde{S}}\right)^{-1/2} \exp\left\{-\frac{1}{2}\,\frac{(\tilde{S} - E[\tilde{S}])^2}{\sigma^2_{\tilde{S}}}\right\} \tag{23}$$

and

$$p(r) = \left[2\pi\left(\frac{\sigma^2_{\tilde{S}}}{S^2}\right)\right]^{-1/2} \exp\left\{-\frac{1}{2}\,\frac{(r-E(r))^2}{\frac{\sigma^2_{\tilde{S}}}{S^2}}\right\} \tag{24}$$

The square root of the average value of the squared relative error, $\sqrt{E(r^2)}$, can be used as a suitable measure of the error in estimating S. A computer program to evaluate $E(\tilde{S})$, $E(r)$, σ_r, and $\sqrt{E(r^2)} = \sqrt{E^2(r) + \sigma_r^2}$ as specified in Eqs. (18) - (22) for any set of values of m, σ_n, S, A, and B has been prepared.*

Consideration of data obtained through employment of the computer program resulted in the following observations.

1. Consideration should be given to utilization of an A/D converter having logarithmically spaced outputs in radar systems of this type. For example, it was found that for ranges of m, σ_n, S, A, and B pertinent to this radar system smaller values of $E(r^2)$ result if the A/D converter employs 7 logarithmically spaced quantization levels (7 bits) rather than 32 equally spaced output levels (5 bits) because of the greater dynamic range of the logarithmically spaced output A/D converter. It should be noted that at the 4 mc sampling rate employed a 7 level logarithmically spaced output A/D converter is probably simpler to construct than the latter 5 bit 32 level device.

2. Reducing the number of bits employed in the A/D converter to less than 5 bits in order to make feasible the utilization of a higher sampling frequency (at present

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* Acknowledgement is made to R. Carroll for preparation of these programs

it is very difficult to construct A/D converters of greater than 4 mc sampling frequency which contain large numbers of quantization levels) results in an increase in the value of $E(r^2)$ for the range of m, σ_n, S, A, and B relevant to this radar system. The effect of decreasing the number of quantization levels and simultaneously increasing the sampling frequency (and thus increasing m by beating down the 1295mc signals to frequencies greater than 1mc and 2mc) is to increase the component of $E(r^2)$ due to the nonzero average of the relative error E(r) and decrease the component of $E(r^2)$ arising from σ_n.

3. As expected, the component of $E(r^2)$ arising from the nonzero value of E(r) can be made relatively small near values of S corresponding to quantization levels (components of A).

It should be noted that this error analysis presents the errors in S, T, U, V, X, Y, P, and Q rather than the expected errors in the azimuth and elevation signals to the servo system. In order to obtain this latter information, and in order to take into account other factors contributing to errors in aiming the radar antenna such as not completely compensating for Doppler frequency shifts in the received signals, a computer simulation of the system was programmed. The results of the simulation indicated that if Raytheon 4mc 7 bit A/D converters were employed then the radar system inaccuracies resulting from Doppler errors (resulting in an incorrect sampling frequency) would be a significant portion of the overall system inaccuracies and the expected errors in computing the servo signals would be a few percent for the Doppler errors and signal-to-noise ratios present in the radar system.

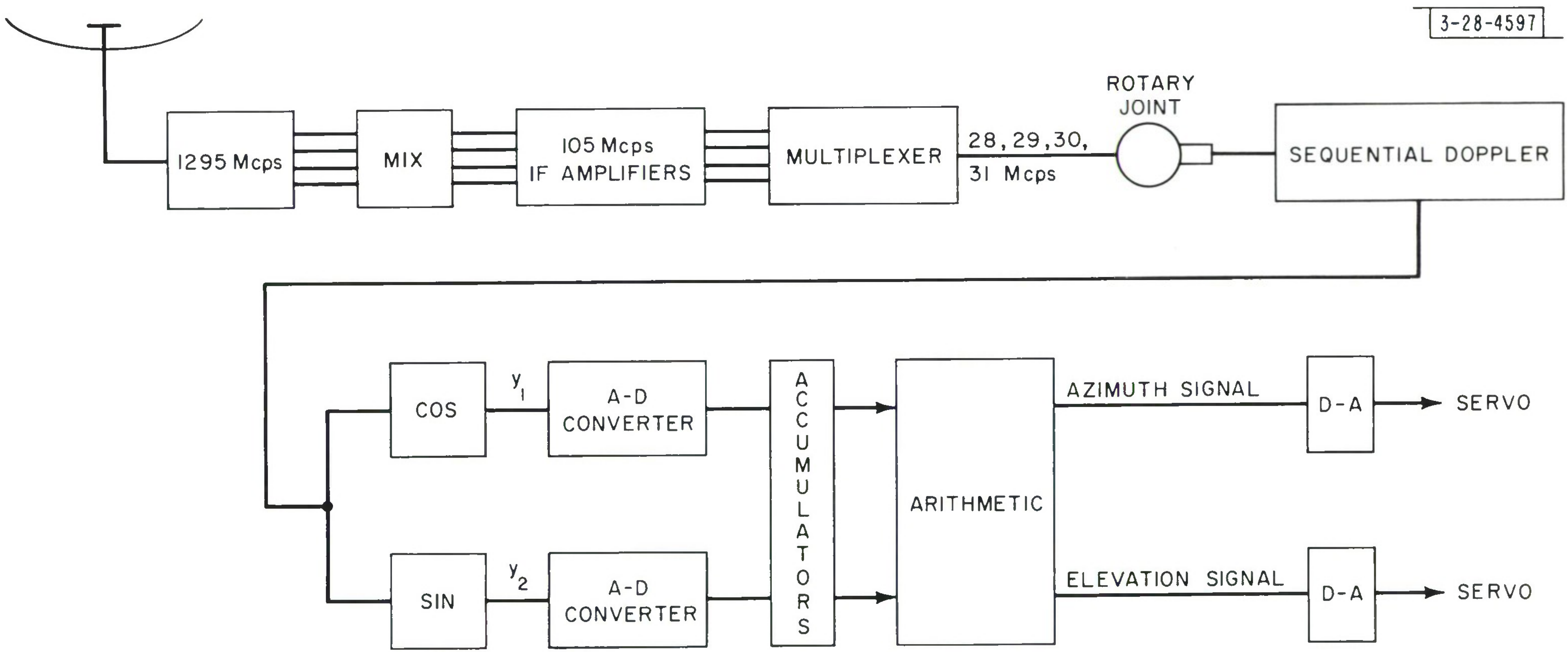
3-28-4597
1295 Mcps
MIX
105 Mcps
IF AMPLIFIERS
MULTIPLEXER
28, 29, 30,
31 Mcps
ROTARY
JOINT
SEQUENTIAL DOPPLER
COS
SIN
y1
y2
A-D
CONVERTER
A-D
CONVERTER
ACCUMULATORS
ARITHMETIC
AZIMUTH SIGNAL
ELEVATION SIGNAL
D-A
D-A
SERVO
SERVO

Fig. 1

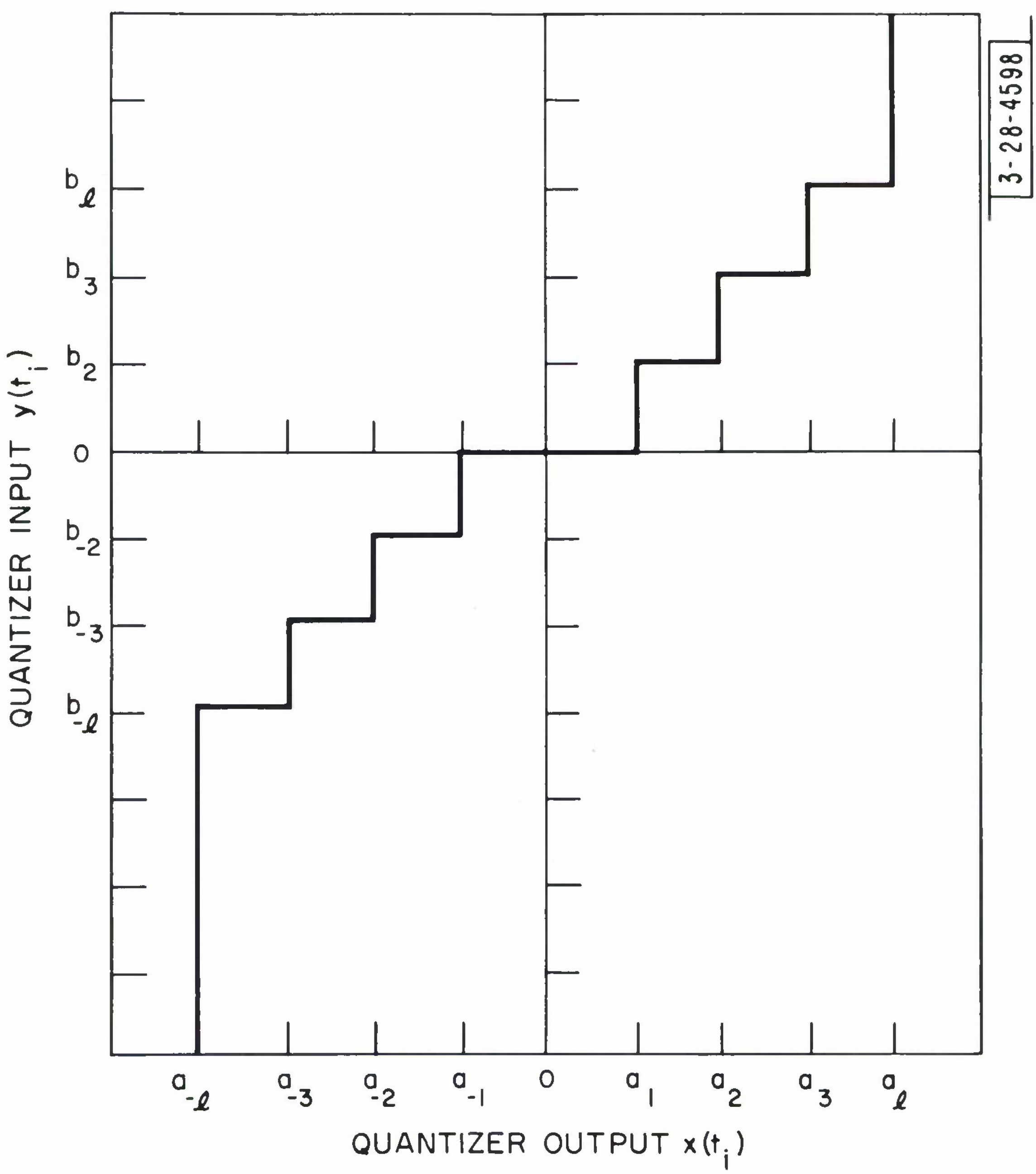
3-28-4598
QUANTIZER INPUT $y(t_i)$
b_ℓ
b_3
b_2
0
b_{-2}
b_{-3}
$b_{-\ell}$
$a_{-\ell}$
a_{-3}
a_{-2}
a_{-1}
0
a_1
a_2
a_3
a_ℓ
QUANTIZER OUTPUT $x(t_i)$

Fig. 2

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